Statistical mechanics of the multi-constraint continuous knapsack problem

1997 J. Phys. A: Math. Gen. 30 1047

(http://iopscience.iop.org/0305-4470/30/4/008)

J. Phys. A: Math. Gen. **30** (1997) 1047–1058. Printed in the UK PII: S0305-4470(97)74146-5

Statistical mechanics of the multi-constraint continuous knapsack problem

Jun-ichi Inoue
Department of Physics, Tokyo Institute of Technology, Oh-okayama, Meguro-ku, Tokyo 152, Japan

Received 22 April 1996, in final form 8 July 1996

Abstract. We apply the replica analysis established by Gardner to the multi-constraint continuous knapsack problem, which is one of the linear programming problems and a most fundamental problem in the field of operations research (OR). For a large problem size, we analyse the space of solution and its volume, and estimate the optimal number of items to go into the knapsack as a function of the number of constraints. We study the stability of the replica symmetric (RS) solution and find that the RS calculation cannot estimate the optimal number of items in the knapsack correctly if many constraints are required. In order to obtain a consistent solution in the RS region, we try the zero-entropy approximation for this continuous solution space and get a stable solution within the RS ansatz. On the other hand, in the replica symmetry breaking (RSB) region, the one-step RSB solution is found by Parisi's scheme. It turns out that this problem is closely related to the problem of optimal storage capacity and of generalization by maximum-stability rule of a spherical perceptron.

1. Introduction

Recently, Korutcheva *et al* [1] pointed out that statistical mechanical analysis based on the replica calculation [2] can be used to investigate some specific statistical properties of optimal solutions for an optimization problem, the so-called knapsack problem. The knapsack problem they studied is one of the integer programming problems in which the variables are all integer. The problem is stated as follows. Let us suppose that a man climbs a mountain. He put N items, $s_1, s_2, \ldots, s_N$ in his knapsack; each item has its own weight, $a_1, a_2, \ldots, a_N$, and 'utility', $c_1, c_2, \ldots, c_N$ which means, for example, its own 'price' or 'necessity in climbing a mountain'. He cannot bring all the N items in his knapsack and he has to leave some of them, and must decide which combinations of items is best for him. We can write down the above situation as

$$\max\left\{\sum_{j=1}^{N} c_j s_j \,|\, \sum_{j=1}^{N} a_j s_j \leqslant b, s_j \in \{0,1\}, j=1,\ldots N\right\} \tag{1}$$

where the decision variable $s_j = 1$ is defined such that the jth item goes into the knapsack, and $s_j = 0$ otherwise. The constant b represent the weight limit. If we have to take account not only of the strength of the man but also the capacity of the knapsack itself (i.e. the volume of the knapsack is limited), this new constraint must be considered and the problem become more complex. Thus it is meaningful for us to generalize our problem to a new form with K constraints as follows:

$$\max\left\{\sum_{j=1}^{N} c_j s_j \,|\, \sum_{j=1}^{N} a_{kj} s_j \leqslant b_k, k=1,\ldots,K, s_j \in \{0,1\}, j=1,\ldots,N\right\}. \tag{2}$$

Following Korutcheva *et al* [1], we set all the utilities c_j to $\frac{1}{2}$, and we set all the limits b_k, representing capacities of the man, of his knapsack, etc, to b.

In order to treat the multi-constraint ($K \sim N$) knapsack problem in a more familiar way for physicists, we convert the decision variables $s_j = (1, 0)$ into spin variables $\hat{s}_j = (1, -1)$ by transformation $\hat{s}_j = 2(s_j - \frac{1}{2})$. The load, a_{kj}, is also transformed as $a_{kj} = \frac{1}{2} + \xi_{kj}$, where the quenched random variable, ξ_{kj}, has zero mean and variance, σ^2, and is assumed to obey the distribution

$$P(\xi_{ki}) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\xi_{ki})^2}{2\sigma^2}\right). \tag{3}$$

As loads a_{ki}, which means 'weights' for example, should be positive, we must choose the above variance, σ^2, small enough and for this small variance, the fraction to items for which $a_{ki} < 0$ for all k vanishes exponentially with increasing K. In this paper, we use $\sigma = \frac{1}{12}$ following Korutcheva *et al* [1]. Using these transformations, our problem is rewritten as follows.

$$\max\left\{U = \frac{N}{4} + \frac{1}{2}\sum_{j=1}^{N} \hat{s}_j | Y(\{\boldsymbol{s}\}, \{a_{ki}\}, b) \leqslant 0, k = 1, \ldots, K, \hat{s}_j \in \{1, -1\}, j = 1, \ldots, N\right\} \tag{4}$$

where

$$Y(\{\boldsymbol{s}\}, \{a_{ki}\}, b) = \frac{1}{2}\sum_{j=1}^{N}(1 + \hat{s}_j)\xi_{kj} + = \frac{1}{4}\sum_{j=1}^{N}\hat{s}_j + \frac{N}{4} - b. \tag{5}$$

From the constraints appearing in equations (4) and (5), if we treat the case of $N/4 \ll b$, we can put most of the items into the knapsack. On the other hand, the number of items to be in the knapsack is too small when we set $n/4 \gg b$. For this reason, we treat the case of $N/4 = b$ which means that about half of the items go into the knapsack.

The case of finite K in the limit N infinity was investigated by Meanti *et al* [3] by the Langrangian relaxation technique. They gave the explicit results for the special cases of $K = 1$ and $K = 2$. Recently, Fontanari [4] presented an explicit calculation of the annealed entropy of the configuration $\boldsymbol{s}^*$ which satisfies K constraints and gives the total benefit $E = \sum_i s_i^*/2$. He estimated the upper limit of the total benefit using the zero-entropy condition.

He compared it with the exact result from Langrangian relaxation for the case of $K = 1, 2$ (exact solutions) and $K > 2$ to conclude that the annealed approximation becomes good as K increases.

The original knapsack problem consists of only one constraint, or at most, several constraints. One may feel that only such a case is worth investigating. However, in the actual operations research field, we sometimes face multi-constraint ($K \sim N$) knapsack problems [5], an example of which is investigated in this paper.

If we look at the above problem from an actual operations research (OR) point of view, difficulties lie in the discreteness of knapsack variables. When hardship of this sort confronts us, it is customary to apply the 'linear programming relaxation' technique [5]. Linear programming relaxation is an approach to solve the integer programming problem approximately, which brings us to the neighbourhood of the exact solution. Actually, for linear programming problems, a lot of useful methods, for example, the simplex method or the interior point method, have proposed and improved [5].

In addition, it is worthwhile to investigate the objective function for a linear programming problem itself (not as a relaxation of the integer programming problem such as the original knapsack problem) [5]. The linear programming problem is defined as follows:

$$\max\left\{\frac{1}{2}\sum_{j=1}^{N}\hat{s}_j \,|\, \sum_{j=1}^{N}a_{kj}\hat{s}_j \leqslant b, k=1,\ldots,K, 0\leqslant \hat{s}_j \leqslant \infty, j=1,\ldots,N\right\}. \tag{6}$$

Problems of the above style occur very often in our life—'The diet (nutrition) problem' is one of them. The diet problem asks us to determine the intake $\{s_j\}$ of each food, s_j, which has its own nutrients, a_{kj}, i.e. calcium, protein, vitamin, energy, etc, and one has to take each nutrient within the limit, i.e. $\sum_j a_{kj} \leqslant b(k=1,\ldots,K)$, where a_{kj} is kth nutrient of the jth food. We should notice that valuables in this problem are real numbers, not integers, and the linear programming problem has more applications than the integer programming problem [5]. For two reasons, it is worth investigating the multi-constraint $(K \sim N)$ continuous knapsack problem.

When we look at our knapsack problem from the stand-point of linear programming relaxation, the knapsack valuable, s_i, which has a real value and determines whether the ith item goes into the knapsack, takes any value in the real subspace and satisfies $\sum_i s_i^2 = N$. We treat the next real-variable problem in the present paper

$$\max\left\{U=\frac{N}{4}+\frac{1}{2}\sum_{j=1}^{N}\hat{s}_j | Y(\{\boldsymbol{s}\},\{a_{ki}\},b) \leqslant 0, k=1,\ldots,K, \hat{s}_j\in\{-\infty,+\infty\}, j=1,\ldots,N\right\} \tag{7}$$

and equation (5). Although this relaxation of constraint $\sum_i \hat{s}_i^2 = N$ and $\hat{s}_i \in \{-\infty,+\infty\}$ in equation (7) somewhat obscures the direct significance of the variable s_i, the global macroscopic behaviour of the system is expected not to be affected by this approximation as was mentioned above. And this constraint could be justified as simply the spherical version of the problem investigated by Korutcheva *et al* [1].

Using this linear programming relaxation technique, we can obtain the approximate configuration $s^* = (s_1^*,\ldots,s_N^*)$ and we get a higher value of the objective function (or the total profit)

$$E = \tfrac{1}{2}\sum_{j=1}^{N} s_j. \tag{8}$$

In this paper, we investigate the knapsack problem by the linear programming relaxation technique and estimate the objective function when the number of constraints is given for a large problem size.

2. Replica symmetric theory

For the parameter regions mentioned in the previous section, we consider the case with continuous variables under the normalization constraint $\sum_{i=1}^{N}(\hat{s}_i)^2 = N$. For a large problem size, the fractional volume of solution $V(M,\{\xi_{ki}\})$ to equation (7) is written by introducing the 'fluctuation of the magnetization' around 0, which is of the order of $\mathcal{O}(1)$,

$$M = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\hat{s}_i \tag{9}$$

as

$$V(M,\{\xi_{ki}\}) = \left[\prod_{k=1}^{K}\int \mathrm{d}\hat{s}_i\, \Theta\left(-\left(\sum_{i=1}^{N}\frac{(1+\hat{s}_i)\xi_{ki}}{\sqrt{N}}+\frac{M}{2}\right)\right)\delta\left(\sum_{i=1}^{N}(\hat{s}_i)^2-N\right)\right.$$
$$\left.\times\delta\left(\sqrt{N}M-\sum_{i=1}^{N}\hat{s}_i\right)\right]\bigg/\left[\prod_{k=1}^{K}\int \mathrm{d}\hat{s}_i\,\delta\left(\sum_{i=1}^{N}(\hat{s}_i)^2-N\right)\right]. \qquad (10)$$

This expression is similar to Gardner's volume of a binary spherical perceptron with local field, $-\sum_i \hat{s}_i\xi_{ki}/\sqrt{N}$, and stability criterion constant, $-M/2$, appearing in the problem of optimal capacity [6, 7] or of the generalization of maximum-stability rule [8]. Thus it is possible to use the same technique; and this V is the value of configuration $\{\hat{s}_i\}$ which has a fixed M, implying also a fixed utility U. The typical of V is given by $\exp(\langle\langle\log V\rangle\rangle)$, where $\langle\langle\cdots\rangle\rangle$ denotes the averaging over quenched disorder. We perform the average of $\log V$ with the distribution (3) over the different sets of solutions using the replica trick as follows.

$$\langle\langle\log V\rangle\rangle = \lim_{n\to 0}\frac{\langle\langle V^n\rangle\rangle-1}{n}. \qquad (11)$$

The averaging of the power $\langle\langle V^n\rangle\rangle$ is accomplished by introducing an ensemble of n identical replicas

$$\langle\langle V^n\rangle\rangle = \left\langle\!\!\left\langle\prod_{\alpha=1}^{n}\left[\prod_{k=1}^{K}\int \mathrm{d}\hat{s}_i^\alpha\,\Theta\left(-\left(\sum_{i=1}^{N}\frac{(1+\hat{s}_i^\alpha)\xi_{ki}}{\sqrt{N}}+\frac{M}{2}\right)\right)\delta\left(\sum_{i=1}^{N}(\hat{s}_i^\alpha)^2-N\right)\right.\right.\right.$$
$$\left.\left.\left.\times\delta\left(\sqrt{N}M-\sum_{i=1}^{N}\hat{s}_i^\alpha\right)\right]\bigg/\left[\prod_{k=1}^{K}\int \mathrm{d}\hat{s}_i^\alpha\,\delta\left(\sum_{i=1}^{N}(\hat{s}_i^\alpha)^2-N\right)\right]\right\rangle\!\!\right\rangle. \qquad (12)$$

Then we introduce the order parameter which is the overlap between two solutions labelled by two replica indices α and β,

$$q_{\alpha\beta} = \frac{1}{N}\sum_i \hat{s}_i^\alpha\hat{s}_i^\beta. \qquad (13)$$

By the standard procedure, we obtain within the replica symmetric (RS) ansatz

$$\langle\langle V^n\rangle\rangle = \exp\left[Nn\left\{\mathrm{extr}_{q,\hat{E},\hat{q},\hat{M}}G(q,\hat{E},\hat{q},\hat{M})+\mathcal{O}\left(\frac{1}{N}\right)\right\}\right] \qquad (14)$$

where $\mathrm{extr}_{q,\hat{E},\hat{q},\hat{M}}$ denotes the extremization with respect to the parameters q, $\hat{E}$, $\hat{q}$ and $\hat{M}$. Here

$$G = \alpha G_1 + G_2 - \frac{\mathrm{i}}{2}\hat{q}q + \mathrm{i}\hat{E} \qquad (15)$$

$$G_1 = \log\prod_\alpha\int_{-M/2}^{\infty}\frac{\mathrm{d}\lambda_\alpha}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}x_\alpha\,\exp\left[\mathrm{i}\sum_\alpha\lambda_\alpha-\sigma^2\sum_\alpha(x_\alpha)^2-\sigma^2(1+q)\sum_\beta\sum_{\alpha<\beta}x_\alpha x_\beta\right] \qquad (16)$$

$$G_2 = \log\int_{-\infty}^{\infty}\prod_\alpha \mathrm{d}\hat{s}^\alpha\,\exp\left[-\mathrm{i}\hat{M}\sum_\alpha\hat{s}^\alpha-\mathrm{i}\hat{q}\sum_\beta\sum_{\alpha<\beta}\hat{s}^\alpha\hat{s}^\beta-\mathrm{i}\hat{E}\sum_\alpha(\hat{s}^\alpha)^2\right]. \qquad (17)$$

Using the saddle-point equation with respect to the $\hat{M}$, we obtain

$$\hat{M} = 0$$

Using this result and the saddle-point equation with respect to $\hat{E}$, we get

$$G(M,q) = \alpha\int_{-\infty}^{\infty}\mathrm{D}t\,\log H\left(\frac{M/(2\sigma)-t\sqrt{1+q}}{\sqrt{1-q}}\right)+\frac{1}{2}\log(1-q)+\frac{1}{2}\frac{q}{1-q} \qquad (18)$$

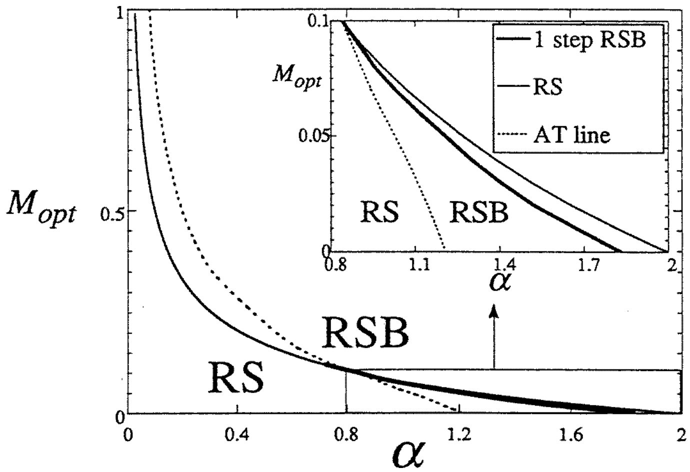

Figure 1. $M_{\rm opt} - \alpha$ line calculated by the replica symmetry theory, AT line and one-step replica symmetry-breaking solution. In each line, we set $\sigma = \frac{1}{12}$. The replica symmetric solution is stable above $M \sim 0.100$. In this figure, the region where the replica symmetry is broken is shown enlarged. The optimal profit $M_{\rm opt}$ decreases slightly by the one-step RSB calculation.

where

$$\mathrm{D}t \equiv \frac{\exp(-t^2/2)}{\sqrt{2\pi}}\mathrm{d}t \tag{19}$$

and

$$H(x) \equiv \int_x^\infty \mathrm{D}t. \tag{20}$$

Finally we estimate the saddle point of equation (18) with respect to the q in the limit $q \to 1$. This means that only one optimal solution (which satisfies all $K = \alpha N$ conditions in equation (7)) is selected for a given number of constraints in this limit and then optimal $M_{\rm opt}$ can be calculated by $q \to 1$ for this optimal $\hat{s}$ which satisfies all K constraints (7). Thus we obtain $M_{\rm opt}$ as a function of $\alpha = K/N$. This result is shown in figure 1.

From this result, we see that $M - \alpha$ lines show the next behaviour. As the number of constraints increases, M decreases and as a result, total utilities decrease (we should remember that total utilities are given as $U = N/4 + \sum_{j=1}^N \hat{s}_j/2$ and $M = \sum_{i=1}^n \hat{s}_i/\sqrt{N}$). In figure 2, we also plotted the optimal profit $M_{\rm opt}$ for the case of Ising variables using the zero-entropy condition of solution discrete space taken from [1]. This result shows that we can obtain a larger optimal profit than that of the original knapsack problem by our relaxation of variables.

3. Three relevant lines

3.1. The AT line

In order to investigate the local stability of the RS solution, we follow the usual Almeida and Thouless [9] argument and introduce the fluctuations of the order parameters around

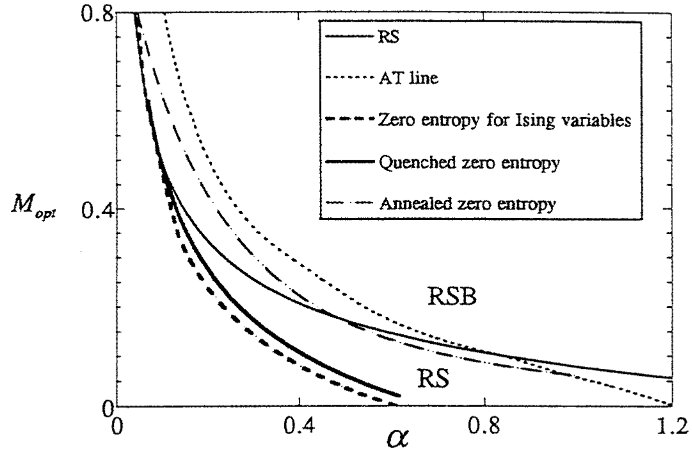

Figure 2. The zero-entropy lines (quenched and annealed calculations). The quenched zero-entropy line is stable in the RS region. The optimal profit for Ising variables [1] is also plotted. This line is very close to the quenched zero-entropy line.

the RS order parameter as follows:

$$\begin{aligned} q_{\alpha\beta} &= q + \delta q_{\alpha\beta} \\ \hat{q}_{\alpha\beta} &= \hat{q} + \delta \hat{q}_{\alpha\beta} \\ M_\alpha &= M + \delta M_\alpha \\ \hat{M}_\alpha &= \hat{M} + \delta \hat{M}_\alpha \\ \hat{E}_\alpha &= \hat{E} + \delta \hat{E}_\alpha . \end{aligned} \tag{21}$$

It turns out that only fluctuations in $q_{\alpha\beta}$ and $\hat{q}_{\alpha\beta}$ lead to instability. The function to be investigated has the form

$$\alpha G_1(q_\alpha, M_\alpha) + G_2(\hat{q}_{\alpha\beta}, \hat{M}_\alpha, \hat{E}_\alpha) + \mathrm{i} \sum_\beta \sum_{\alpha<\beta} q_{\alpha\beta} \hat{q}_{\alpha\beta}. \tag{22}$$

From this expression, we can apply Gardner's [6, 7] analysis to our case and obtained the AT line as

$$\alpha \left[\int_{-\infty}^{\infty} \mathrm{D}t \left\{ 1 - \frac{\int_{I_z} z^2\, \mathrm{D}z}{\int_{I_z} \mathrm{D}z} + \left(\frac{\int_{I_z} z\, \mathrm{D}z}{\int_{I_z} \mathrm{D}z} \right)^2 \right\} \right]^2 < 1 \tag{23}$$

where

$$\int_{I_z} \equiv \int_{(M/(2\sigma)-\sqrt{1+qt})/\sqrt{1-q}}^{\infty} \tag{24}$$

and q is the order parameter obtained by the RS calculation and is given as

$$q = \alpha \frac{\sqrt{1-q}}{\sqrt{1+q}} \int \mathrm{D}t \left(\frac{M}{2\sigma} \sqrt{1+q} - 2t \right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(M/(2\sigma) - t\sqrt{1+q})^2}{2(1-q)} \right) \Bigg/ H\left(\frac{M/(2\sigma) - t\sqrt{1+q}}{\sqrt{1-q}} \right). \tag{25}$$

We plot this condition, equation (23), in figure 1.

From this figure, we see that the replica symmetry is stable for $M > M_{\mathrm{AT}} = 0.100$ ($\alpha < \alpha_{\mathrm{AT}} = 0.846$). However, when M becomes smaller than this critical value, the replica symmetry breaking (RSB) occurs and the RS saddle point becomes unstable. The AT point ($\alpha_{\rm AT}(M_{\rm AT})$) is a critical point where the RS order parameter $q_{\rm RS} = q$ split into (one-step) RSB saddle point represented by q_0 and q_1. If this RSB transition at $\alpha_{\rm AT}$ is continuous, the AT condition, equation (23), gives a correct criterion of RSB. On the other hand, if the RSB transition is discontinuous (first order), we should regard this discontinuity point as the symmetry breaking point rather than the AT stability limit. Fortunately, as we see in the next section, this transition is continuous and the AT argument is correct.

In order to understand this situation physically, we can investigate the RSB from the disconnectivity of the solution space, $\hat{s}$, according to Monasson and O'Kane [10], who treated RSB of perceptron with nonmonotonic transfer function.

The solution condition, equation (7), can be rewritten as

$$\sum_i \hat{s}_i \frac{\xi_{ki}}{\sqrt{N}} \leqslant -\frac{M}{2} + \sum_i \frac{\xi_{ki}}{\sqrt{N}} = -\frac{M}{2} + \mathcal{O}(1).$$

Then if M is positive, for many constraints K, the space of $\hat{s}$ which satisfies equation (7) as well as the normalization constraint $\sum(\hat{s})^2 = N$ consists of a single domain and the solution space is connected. On the other hand, as M become negative, the space of $\hat{\boldsymbol{s}}$ splits into a lot of domains and the solution space is disconnected.

Here we can estimate the $\mathcal{O}(1)$ term roughly appearing on the right-hand side of the above inequality as $\sum_j \xi_j^\mu/\sqrt{N} \sim (1/\sqrt{N}) \times \sqrt{N}\sigma = \frac{1}{12}$. We should notice that this term is of $\mathcal{O}(1)$ and we used the standard deviation $\sigma = \frac{1}{12}$. Therefore, if we investigate the disconnectivity of solution space, replica symmetry must be broken for $M < \frac{1}{6} = 0.1667$. This value is not so far from the value obtained by the AT argument. We conclude that this disconnectivity leads to RSB. For this RSB region, we will later find the one-step RSB solution following the scheme of Parisi [11].

3.2. Zero entropy line

In the previous subsection, we calculated the volume of solution space $\exp(\log\langle\langle V\rangle\rangle)$ for quenched random loads ξ_{ki}. For optimal $M_{\rm opt}$, this volume shrinks to zero continuously.

Thus the increase in the number of constraints leads to a decrease of the solution space, $\hat{\boldsymbol{s}}$, in a discrete manner, and the entropy of solution space is $S(M_{\rm opt}) = 0$ at $M = M_{\rm opt}$. Following the idea suggested by Krauth and Mézard [12] we expect that the optimal M for continuous variables may be obtained when the volume of the solution typically contains a single point of hypercube. From analogy with the discrete-variable problem, this should occur around

$$V \sim \frac{1}{2^N}. \tag{26}$$

3.2.1. Annealed calculation. We first calculate the condition (26) by the annealed approximation as

$$\log\langle\langle V\rangle\rangle = G - H = -\log 2. \tag{27}$$

This condition is easily calculated by

$$G = \alpha \log H\left(\frac{M}{2\sqrt{2}}\right) + \frac{1}{2}(1 + \log(2\pi)) \tag{28}$$

$$H = \tfrac{1}{2}(1 + \log(2\pi)). \tag{29}$$

Finally we get

$$\alpha \log H\left(\frac{M}{\sqrt{2}}\right) = -\log 2. \tag{30}$$

This is shown in figure 2.

3.2.2. Quenched calculation. Next we calculate the above-zero entropy condition (26) by the quenched calculation defined as

$$\langle\langle \log V \rangle\rangle = \frac{1}{2N}. \tag{31}$$

This leads to

$$\alpha \int \mathrm{D}t \, \log H\left(\frac{M/2\sigma - t\sqrt{1+q}}{\sqrt{1-q}}\right) + \frac{1}{2}\frac{q}{1-q} + \frac{1}{2}\log(1-q) = -\log 2. \tag{32}$$

We also plotted this result in figure 2.

From figure 2 we find that this line, equation (32), obtained by the quenched zero entropy condition is stable in the RS region. In figure 4 we plotted the behaviour of the order parameter q as a function of α which gives zero entropy and satisfies equation (32). From these results, we may conclude that our zero-entropy approach gives a meaningful and consistent criterion for the optimal M_{opt}.

3.3. The one-step RSB solution

In this subsection, we calculate the one-step RSB solution following Parisi [11, 13]. We try to find a first candidate for matrices $\boldsymbol{q}$ and $\hat{\boldsymbol{q}}$ appearing in the free energy. Using Parisi's suggestion, we divide the n replicas into m/n groups of m replicas. Next we set $q_{\alpha\beta} = q_1$, $\hat{q}_{\alpha\beta} = \hat{q}_1$, if α and β belong to the same group, and $q_{\alpha\beta} = q_0$, $\hat{q}_{\alpha\beta} = \hat{q}_0$, is α and β belong to different groups and we set $q_{\alpha\alpha} = 0$. We express $\boldsymbol{q}$ and $\hat{\boldsymbol{q}}$ in terms of a tensor product,

$$\boldsymbol{q} = (q_1 - q_0)\mathbf{1}_{n/m} \otimes \boldsymbol{e}_m \boldsymbol{e}_m^T + q_0 \boldsymbol{e}_n \boldsymbol{e}_n^T \tag{33}$$

$$\hat{\boldsymbol{q}} = (\hat{q}_1 - \hat{q}_0)\mathbf{1}_{n/m} \otimes \boldsymbol{e}_m \boldsymbol{e}_m^T + \hat{q}_0 \boldsymbol{e}_n \boldsymbol{e}_n^T \tag{34}$$

where $\mathbf{1}_k$ is a k-dimensional unit matrix and $\boldsymbol{e}_k^T = (1, \ldots, 1)$. Using this form of broken symmetry of replica, we get the free energy as follows,

$$\begin{aligned} f_{\mathrm{1RSB}}(q_1, q_0, m) = {} & \frac{\alpha}{m} \int \mathrm{D}y \, \log \int \mathrm{D}z \left\{ H\left(\frac{M/(2\sigma) - y\sqrt{1+q_0} - z\sqrt{q_1 - q_0}}{\sqrt{1-q_1}}\right)\right\}^m \\ & + \frac{1}{2m}\log[1 - q_1 + m(q_1 - q_0)] + \frac{1}{2}\left(1 - \frac{1}{m}\right)\log(1 - q_1) \\ & + \frac{q_0}{2[1 - q_1 + m(q_1 - q_0)]} \end{aligned} \tag{35}$$

where we used the saddle-point equations with respect to $\hat{q}_1$, $\hat{q}_0$, $\hat{E}$ and $\hat{M}$.

Before calculating the one-step RSB solution, we had better investigate whether it actually exists or not by minimizing the one-step RSB free energy, f_{1RSB}, with respect to the order parameters q_0, q_1 and m. For example, we try to investigate the case of $M_{\mathrm{opt}} = 0$. Varying α in the free energy f_{1RSB}, we look for the set of order parameters q_0, q_1 and m to minimize f_{1RSB}. In table 1, we list the result for the typical value of α. From this table, we see that until $\alpha \sim \alpha_{\mathrm{AT}} = 1.20$, the replica symmetric saddle point (which is $q_0 = q_1 = q$ and is independent of m) is stable and the one-step RSB solution does not exist. When

Table 1. The RS order parameter q and the one-step RSB order parameters (q_0, q_1, m) are listed with the free energies for several values of α. Until $\alpha \sim \alpha_{\rm AT} = 1.20$ the RS saddle point is stable and the one-step RSB solution does not exist. Above the $\alpha_{\rm AT}$, the free energy of the one-step RSB approximation is lower than that of RS approximation.

α	$q_{\rm RS}$	q_0	q_1	m	$f_{\rm RS}$	$f_{\rm RSB}$
0.90	0.7129	—	—	—	−1.5336	—
1.00	0.7773	—	—	—	−1.9346	—
1.10	0.8408	—	—	—	−2.4859	—
1.20	0.8901	—	—	—	−3.3794	—
1.25	0.9073	0.8213	0.9901	0.1688	−3.7642	−5.7847
1.30	0.9211	0.8155	0.9902	0.1858	−4.4110	−6.0763

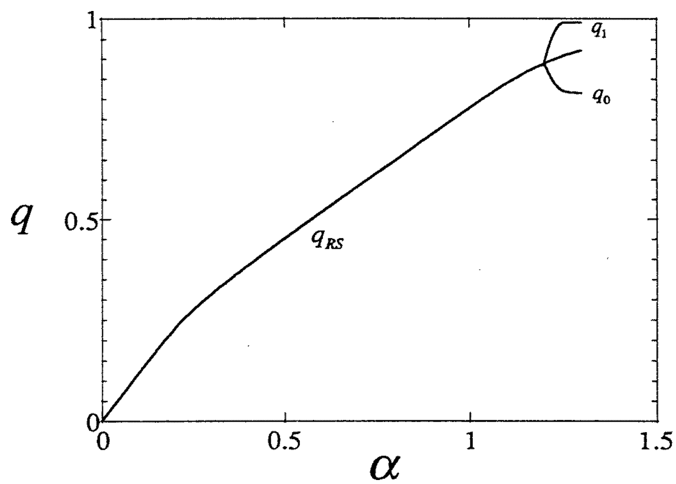

Figure 3. Order parameter q as a function of the relative number of constraints α for the case of $M_{\rm opt} = 0$. RS order parameter q splits into the one-step RSB order parameter (q_0, q_1) continuously at $\alpha_{\rm AT} \sim 1.20$.

we exceed the critical value $\alpha_{\rm AT}$, the replica symmetric saddle point becomes unstable and order parameter q splits into one-step RSB order parameters q_0 and q_1 continuously as also shown in figure 3. From this continuous transition, we conclude that the AT argument in the previous section is consistent. Comparing the free energy $f_{\rm RS}$ and $f_{\rm 1RSB}$, we see that the global minimum of this system goes from the RS saddle point q to the one-step RSB saddle point (q_0, q_1) at the AT instability point. Therefore, the one-step RSB solution actually exists.

In order to obtain the solution in the limit $q_1 \to 1$, we use the scaling $q_1 = 1 - \epsilon$, $q_0 = Q$ and $m = \mu\epsilon$ to get

$$F_{\rm 1RSB}(Q, \mu, \epsilon) = \frac{1}{\epsilon} f_{\rm 1RSB}(Q, \mu) + O(\log \epsilon). \tag{36}$$

Here

$$f_{\rm 1RSB}(Q, \mu) = \frac{\alpha}{\mu} \int {\rm D}y \, \log[H_1 + H_2] + \frac{1}{2\mu} \log[1 + \mu(1 - Q)] + \frac{Q}{2[1 + \mu(1 - Q)]} \tag{37}$$

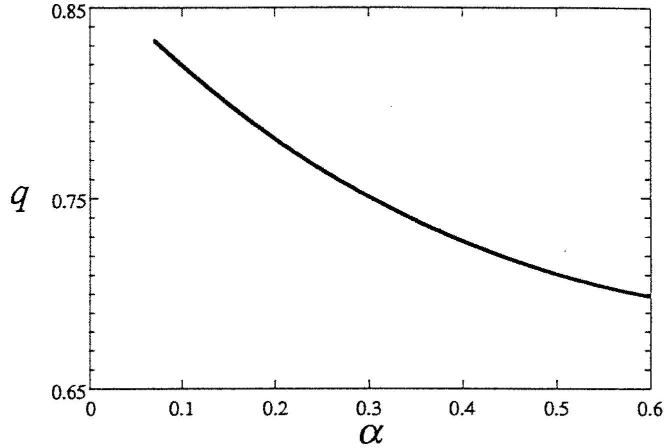

Figure 4. Overlap q which satisfies the zero-entropy condition.

where

$$H_1 \equiv H\left(\frac{M/(2\sigma) - y\sqrt{1+Q}}{\sqrt{1-Q}}\right) \tag{38}$$

and

$$H_2 \equiv \frac{\exp\left(-\frac{\mu(M/(2\sigma)-y\sqrt{1+Q})^2}{2[1+\mu(1-Q)]}\right)}{\sqrt{1+\mu(1-Q)}} H\left(\frac{M/(2\sigma) - y\sqrt{1+Q}}{\sqrt{1-Q}\sqrt{1+\mu(1-Q)}}\right). \tag{39}$$

The saddle-point equations with respect to μ, Q and ϵ

$$\left.\frac{\partial F}{\partial \mu}\right|_{\mu_c, Q_c, \alpha_c} = \left.\frac{\partial F}{\partial Q}\right|_{\mu_c, Q_c, \alpha_c} = F\Big|_{\mu_c, Q_c, \alpha_c} = 0 \tag{40}$$

give the critical values α_c calculated by the one-step RSB scheme. We solved equation (54) numerically and show the results in figure 2.

From this figure, we see that the RSB decreases the optimal $M_{\rm opt}$ slightly. We also confirm this result from the fact that the RSB point $q(\alpha_{\rm AT})$ comes close to 1. In figure 5, we also show the order parameter $q_0 = Q$ as a function of the optimal profit $M_{\rm opt}$. From this figure we see that, as $M_{\rm opt}$ becomes close to 0.100, i.e. as RSB becomes weak, Q becomes 1 gradually ($Q = q_0 = 0.9912$ for $m = 0.100$). This result is reasonable from the meaning of RSB. The reason is that in the critical limit of α, the one-step RSB order parameter q_0 and q_1 become close to $q_{\rm RS} = q$ and $q \to 1$ if replica symmetry does not break down.

In order to confirm that this one-step RSB solution is exact, we must check the stability of the one-step RSB saddle point by the same technique as that used for the AT line and investigate if the free energy of the one-step calculation is lower than that of the second-step calculation. Or the distribution of sizes of the 'disconnected' domains of solution space, s, must be computed analytically by the same technique as in Monasson and O'Kane [10]. This is a highly nontrivial problem. However, we can conjecture that the two-step RSB can hardly decrease the optimal profit even if the two-step RSB solution exists, because the optimal profit of RS calculation and that of one-step RSB calculation are very close to each other as we saw above.

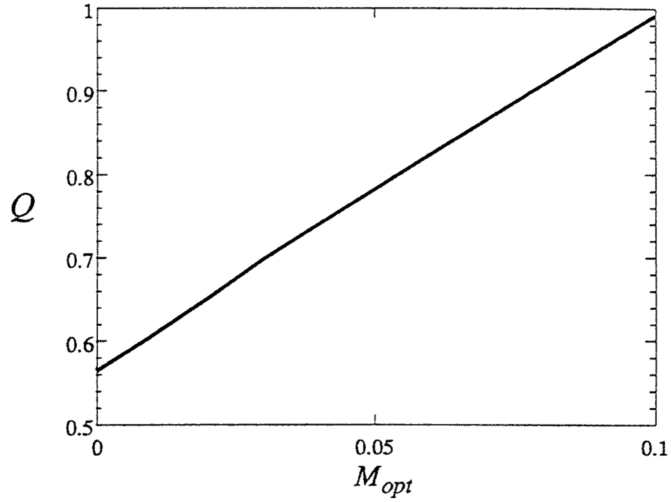

Figure 5. One-step RSB order parameter $q_0 = Q$ as a function of the optimal profit $M_{\rm opt}$ in the limit of $q_1 \to 1$. As RSB becomes weak (as $M_{\rm opt}$ closes to 0.10, $\alpha(M_{\rm opt} = 0.100) = \alpha_{\rm AT}$), Q closes to 1.

4. Summary and discussion

We have calculated the optimal profit, $M_{\rm opt}$, explicitly for the case of continuous knapsack variables and the number of constraints, K, is of the same order with the number of items, N, by RS calculation explicitly. From the AT argument, the RS solution becomes unstable for $M < 0.100$. We also investigated this symmetry-breaking point by minimizing the one-step RSB free energy directly and saw that this transition is continuous and the AT line is valid. For the argument of the problem of the optimal storage capacity of the perceptron with non-monotonic transfer function investigated by Monasson and O'Kane [10], the physical meaning of this RSB can be understood as the disconnectivity of the solution space. Therefore, from the condition of disconnectivity of solution space, we roughly estimated where RSB begins. For the RSB region, it was necessary to try to find a RSB solution according to Parisi's procedure [11, 13]. Thus using Parisi's scheme, we obtained the one-step RSB solution and the result shows that RSB makes the $M_{\rm opt}$ decrease slightly. Because it is not easy to confirm that this one-step RSB calculation is exact, we can roughly expect that, even if we calculate the two-, three- or further-step RSB solution, the optimal profit obtained at these further RSB saddle points should hardly increase. Instead of investigation in this direction, in the RS region, we can find a consistent solution using the zero-entropy approach and this result agrees with the discrete variables case [1] for the most part. This result may also support the validity of their RS calculation. The present statistical mechanical approach is expected to be applicable to the other linear programming problems.

Acknowledgment

I thank Professor H Nishimori for useful discussions during this work.

References

[1] Korutcheva E, Opper M and Lòpez L 1994 *J. Phys. A: Math. Gen.* **27** L645
[2] Kirkpatrick S and Sherrington D 1978 *Phys. Rev.* B **17** 4384
[3] Meanti M, Rinnooy Kan A H G, Stougie L and Vercellis C 1990 *Math. Prog.* **46** 3
[4] Fontanari J F 1995 *J. Phys. A: Math. Gen.* **28** 4751
[5] Chvátal V 1983 *Linear Programming* (San Francisco, CA: Freeman)
[6] Gardner E 1988 *J. Phys. A: Math. Gen.* **21** 271
[7] Gardner E and Derrida B 1988 *J. Phys. A: Math. Gen.* **21** 257
[8] Opper M, Kinzel W, Kleinz J and Nehl R 1990 *J. Phys. A: Math. Gen.* **23** L581
[9] de Almeida J R and Thouless D J 1978 *J. Phys. A: Math. Gen.* **11** 983
[10] Monasson R and O'Kane D 1994 *Europhys. Lett.* **27** 85
[11] Parisi G 1979 *J. Phys. A: Math. Gen.* **13** 1101
[12] Krauth W and Mézard M 1989 *J. Physique* **50** 3057
[13] Mézard M, Parisi G and Virasoro M 1986 *Spin Glass Theory and Beyond* (Singapore: World Scientific)